16. Find the vector and Cartesian equations of the plane passing through the points

(-1, 1, 1) and (1, -1, 1) and perpendicular to the plane X+2y+2Z=5 (mor-07, mar-09, Tune-10).

Answer:

$\vec{a}$=-$\vec{\imath}$+$\vec{k}$; $\vec{b}$ = $\vec{l}$ - $\vec{\jmath}$ + $\vec{t}$; $\vec{v}$ = $\vec{\imath}$ + 2 $\vec{\jmath}$ + 2$\vec{k}$

ஃ Vector equation of the plane

$\vec{r}$ = (1 – S) $\vec{a}$ + $\overrightarrow{sb}$ + t$\vec{v}$

ie $\vec{r}$ = (1-S) $\vec{\imath}$ + $\vec{\jmath}$ + $\vec{k}$) + s($\vec{\imath}$ - $\vec{\jmath}$ + $\vec{k}$) + ($\vec{\imath}$ + 2$\vec{\jmath}$ + 2$\vec{k}$)

Cartesian equation of the plane

$(x_1, y_1, z_1) = (-1, 1, 1)$; $(x_2, y_2, z_2) = (1, -1, 1)$; $(l_1, m_1, l_1) = (1,2,2)$

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \end{vmatrix} = 0$$

ie $$\begin{vmatrix} x+1 & y-1 & z-1 \\ 2 & -2 & 0 \\ 1 & 2 & 2 \end{vmatrix} = 0$$

$\Rightarrow \quad 2x + 2y - 3z + 3 = 0$

17. find the vector and Cartesian equation of the plane containing the line

$\frac{x-2}{2} = \frac{y-2}{3} = \frac{z-1}{-2}$ and passing

through the point (-1, 1, -1)

Answer

$\vec{a}$= (-1, 1, -1) ; $\vec{b}$ = (2, 2, 1) $\vec{v}$ = $2\vec{i} + 3\vec{j} - 2\vec{k}$

$x_1\ y_1\ z_1$ $x_2\ y_2\ z$ $l\,\vec{i} + m\,\vec{j} + n\,\vec{k}$

∴ vector equation of the plane

$\vec{r} = (1-s)\,\vec{a} + s\,\vec{b} + t\,\vec{v}$

ie $\vec{r} = (1-s)\,(-\vec{i} + \vec{j} - \vec{k}) + s\,(2\vec{i} + 2\,\vec{j} + \vec{k}) + t\,(2\,\vec{i} + 3\,\vec{j} - 2\,\vec{k})$

Cartesian equation of the plane

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l & m & n \end{vmatrix} = 0$$

ie

$$\begin{vmatrix} x+1 & y-1 & z+1 \\ 3 & 1 & 2 \\ 2 & 3 & -2 \end{vmatrix}$$

⇨ $8x - 10y - 7z + 11 = 0$

18. Find the vector and cartesion equations of the plane passing through the points (2, 2, 01), (3, 4, 2) and (7, 0, 6) (oct -09)

Ans:

$\vec{a}$ = (2, 2, -1) ; $\vec{b}$ = (3, 4, 2) ; $\vec{c}$ = (7, 0, 6)

x_1 y_1 z_1 x_2 y_2 z_2 x_3 y_3 z_3

Vector equation of the plane

$\vec{r}$ = (1-s-t) $\vec{a}$ + s $\vec{b}$ + t $\vec{c}$

ie $\vec{r}$ = (1- s-t) (2 $\vec{i}$ + 2 $\vec{j}$ - $\vec{k}$) + s (3 $\vec{i}$ + 4$\vec{j}$ + 2 $\vec{k}$) + t (7 $\vec{i}$ + 6 $\vec{k}$)

Cartesian equation of the plane

$x-x_1$	$y - y_1$	$z - z1$	
x_2-x_1	$y_2 - y_1$	$z_2 - z_1$	= 0
x_3-x_1	y_3-y_1	$z_3 - z_1$	

ie

x – 2	y – 2	z + 1
1	2	3
5	-2	7

5 x + 2y = 3z = 17

19. Find the vector and cartesion equation of the plane passing through the points with position vectors

$3\vec{i} + 4\vec{j} + 2\vec{k}$, $2\vec{i} - 2\vec{j} - \vec{k}$ and $7\vec{i} + \vec{k}$

Answer :

$\vec{a} = 3\vec{i} + 4\vec{j} + 2\vec{k}$; $\vec{b} = 2\vec{i} - 2\vec{j} - \vec{k}$; $\vec{c} = 7\vec{i} + \vec{k}$

Vector equeation of the plane

$\vec{r} = (1 - s - t)\vec{a} + \overrightarrow{sb} + t\vec{c}$

ie $\vec{r} = (1 - s - t)(3\vec{i} + 4\vec{j} + 2\vec{k}) + s(2\vec{i} - 2\vec{j} - \vec{k}) + (7\vec{i} + \vec{k})$

Cartesian equation of the plane

(x1, y1, z3) = (3, 4, 2) ; (x2, y2, z2) = (2, -1, 1); (x3, y3, z3) = (7, 0, 1)

$$\begin{vmatrix} X - X_1 & Y - Y_1 & Z - Z_1 \\ X_2 - X_1 & Y_2 - Y_1 & Z_2 - Z_1 \\ X_3 - X_1 & Y_3 - Y_1 & Z_3 - Z_1 \end{vmatrix} = 0$$

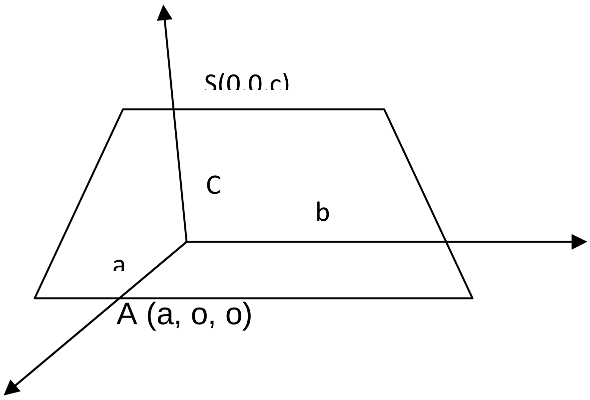
ie
$$\begin{vmatrix} x + 3 & y - 4 & z - 2 \\ -1 & -6 & -3 \\ 4 & -4 & -1 \end{vmatrix} = 0$$

$$6x + 13y - 28z - 14 = 0$$

20. Derive the equation of the plane in the intercept form.

Answer:

Let a, b, c are the x, y and z intercepts respectively



The plane passes through the points (a, o, o), (o, b, o), (o, o, c)

$x_1\ y_1\ z_1 \qquad x_2\ y_2\ z_2 \qquad x_3\ y_3\ z_3$

$\vec{a}$ = a $\vec{c}$; $\vec{b}$ = b $\vec{j}$; $\vec{c}$ = c $\vec{k}$

Vector eqn of the plane is,

$\vec{r}$ = (1-s-t) a$\vec{i}$ + sb$\vec{j}$ + tc$\vec{k}$

ie x $\vec{i}$ + y$\vec{j}$ + z $\vec{k}$ = (1 – s-t) a $\vec{i}$ + sb $\vec{j}$ + tc $\vec{k}$

x = (1-s-t)a ; y = sb ; z = tc

$\frac{x}{a} = 1 - s - t$; $\frac{y}{b} = s$; $\frac{z}{c} = t$

$\frac{x}{a} = 1 - \frac{y}{b} - \frac{z}{c}$

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

This is the intercept form of the plane.

Complex Numbers

1. p represent the variable complex number z

   Find the locus of p, if Im $\frac{2z+1}{iz+1}$ = -2.

   Answer

   let z = x + iy.

   $$\frac{2Z+1}{iZ+1} = \frac{2\,(x+iy)+1}{i\,(x+iy)+1} = \frac{(2x+1)+i2y}{(1-y)+i\,x}$$

   $$= \frac{(2x+1)+i2y}{(1-y)+ix} \times \frac{(1-y)-ix}{(1-y)-ix}$$

   $$= \frac{(2x+1)(1-y)+2xy+i\,(2y\,(1-y)-x\,(2x+1)}{(1-y)2+x2}$$

$$\text{Im}\,\frac{2z+1}{iz+1} = \frac{2y\,(1-y)-x\,(2x+1)}{(1-y)2+x2}$$

But given Im $\frac{2z+1}{i\,z+1}$ = -2

$$\frac{2y\,(1\text{-}y) = X\,(2x+1)}{(1\text{-}y)^2 + x^2} = -2$$

$\Rightarrow$ - x – 2y + 2 = 0.

the locus of p is x + 2y – 2 = 0

2. P represents the variable complex number z, find the locus of P of arg $\left[\frac{z-1}{z+1} = \frac{\pi}{23}\right]$

Answer

Let Z = X + iy

Given arg $\left(\frac{z-1}{z+3}\right) = \frac{\pi}{3}$

arg (z-1) - arg (z + 1) = $\frac{\pi}{23}$

arg (x + iy -1) – arg (x + iy + 1) = $\frac{\pi}{23}$

arg (x-1) +iy) - arg (s+1) + iy = $\frac{\pi}{32}$

tan-1 $\frac{y}{x-1}$ - tan -1 $\frac{y}{x+1} = \frac{\pi}{3}$

$$\tan^{-1}\left(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \left[\frac{y}{x-1}\right]\left[\frac{y}{x+1}\right]} = \frac{\pi}{3}\right)$$

$$\Rightarrow \quad \frac{2y}{x2-1+y2} = \tan\frac{\pi}{3} = \sqrt{3}$$

$\Rightarrow$ $\sqrt{3}x^2 + \sqrt{3}\,y^2 - 2y - \sqrt{3} = 0$

This is the locus of point P.

3. if x and p are the roots of the equation

x2-spn + (P2 +q2) = 0 and tan θ = q

y + p

show that $\frac{(y+\alpha)^n-(y+\beta)^n}{\alpha-\beta} = q^{n-1}\ \frac{sinn\theta}{sin^n\theta}$

Ans

Given x, B are the roots of the equation

x2-2pn + (p2 + q2) = o

ie x2 – (p +iq) + (p – iq) + (p + iq) (p – iq) = o

α = p + iq ; p = p – iq

α – b = 2 q i

Given tanθ = $\frac{q}{y+p}$

y + p = q cotθ ⇒ y = q cotθ - p

y + α = q cotθ + iq ; y + β = q cot θ - iq

= q (cot θ +i) = q (cotθ-i)

y + α = $\frac{q}{sin\theta}$ (Cosθ+isinθ) ; Y + B = $\frac{q}{sin\theta}$ (Cosθ - isino)

$(y + \alpha)^n$ = $\frac{q^n}{sin^n\theta}$ (Cosθ - 2i sinnθ

⇒ $\frac{(y+\alpha)^n-(y+\beta)^n}{\alpha-\ \beta}$ = $q^{n-1}\ \frac{sinn\theta}{sin^n\theta}$

Exercise:

it α and β are the roots of $x^2 - 2x + 2 = 0$ nad

cot θ = y+1 show that $\frac{(y+\alpha)^n - (y+\beta)^n}{\alpha - \beta} = \frac{sinn^\theta}{sin^n\theta}$

4. it α and β are roots of x2 – 2x + 4 = 0, Prove that

αn – pn = I 2 n+1 sin $\frac{\pi}{3}$ and deduct x9 – β9

(sep-06, oct-08, mar-09, mar-12)

Answer

Given α and p are the roots of

x2 – 2x + 4 = 0

ie x2 – (1+I $\sqrt{3}$ + 1 - i$\sqrt{3}$) x + (1 + i$\sqrt{3}$) (I – I $\sqrt{3}$) = 0

α = 1 + I $\sqrt{3}$ & β = 1 – I $\sqrt{3}$

let 1 + I $\sqrt{3}$ = r (Cosθ + isinθ)

a = 1 b = $\sqrt{3}$

r = $\sqrt{a^2}b^2$ = 2

θ = $\tan^{-1} \frac{b}{a} = \tan^{-1} \frac{\sqrt{3}}{1} = \frac{\pi}{3}$

1+i$\sqrt{3}$ = 2 (Cos $\frac{\pi}{3}$ + I sin $\frac{\pi}{3}$)

α = $(1+i\sqrt{3})^n = 2^n$ (Cos $\frac{n\pi}{3}$ + sin $\frac{n\pi}{3}$)

in^y $\beta^n = (1-i\sqrt{3})^n = 2^n$ (Cos $\frac{n\pi}{3}$ - I sin $\frac{n\pi}{3}$)

ie $2^n - \beta^n = 2^n$ (2i sin $\frac{n\pi}{3}$

Substituting n=9 we get,

$\alpha^9 - \beta^9$ = I 2^{10} sin 3π

$\alpha^n - \beta^9$ = 0 sin 3π = 0

5. $x + \frac{1}{x} = 2\cos\theta$, $y + \frac{1}{y} = 2\cos\theta$, Prove that

(i) $\frac{x^m}{y^n} + \frac{y^n}{x^m}$ = 2 cos (mθ - nθ)

(ii) $\frac{x^m}{y^n} - \frac{y^n}{x^m}$ = 2 I sin (mθ - nθ)

Proof Given

$x + \frac{1}{x}$ = 2 Cosθ

x^2 – 2cosθ.x + 1 = 0

x^2 – (cosθ + isin θ + Cos θ - isinθ) x + (Cosθ+ insθ) (cosθ - isnnθ)

x = Cosθ ± isinθ

x we can take x = cosθ + isinθ

iii[by] y = cosθ + isinθ

x^m = cosmθ+isinmθ ; y^n = consθ + isinnθ

$\frac{x^m}{y^n} = \frac{cosm\theta + \text{isinm}\theta}{con\theta + \text{isinn}\theta}$

$\frac{x^m}{y^n}$ = Cos (mθ-nθ) + isin (mθ-nθ)

$\frac{y^n}{x^m}$ = Cos (mθ-nθ) + isin (mθ-nθ)

+ (2) ⇒ $\frac{x^m}{y^n} - \frac{y^n}{x^m}$ = 2 cos (mθ - nθ)

Exercise

it a = cos2x + isin 2x, b = cos 2 β and c = cons2 ϑ+ isin 2 ϑ

Prove that (1) $\sqrt{abc} + \frac{1}{\sqrt{abc}}$ = 2 cos (α + β + ϑ

a = $\frac{1}{2}$ - I $\frac{\sqrt{3}}{2}$ = r (cosθ + isinθ)

a = $\frac{1}{2}$ & b = - $\frac{\sqrt{3}}{2}$

r = $\sqrt{a^2 + b^2}$ = 1

θ = tan-1 $\frac{b}{2}$ = tan-1 (-$\sqrt{3}$ = -$\frac{\pi}{3}$

$\frac{1}{2}$ - I $\frac{\sqrt{3}}{2}$ = Cos (-$\frac{\pi}{3}$)

($\frac{1}{2}$ - $\frac{\sqrt{3}}{2}$) ¾ = $\left[\left(Cos\ (-\frac{\pi}{3} + i \sin(-\frac{\pi}{3}\right)\right]^3$

= (cos (- π) + I sin (-π) ¼

= $(\text{Cos}\ (2\text{k}\,\pi - \ \pi) + isin\ (2\ k\pi - \ \pi\ )^{\frac{1}{4}}$

= Cos (2k-1) $\frac{\pi}{4}$ + isin (2k-1) $\frac{\pi}{24}$; k = 0,1,2,3

Values are cis (-$\frac{\pi}{4}$) ; cis ($\frac{\pi}{4}$) ; cis ($\frac{3\pi}{4}$) ; cis($\frac{5\pi}{4}$)

Their Product = Cis (-$\frac{\pi}{4}$ + $\frac{\pi}{4}$ + $\frac{3\pi}{4}$ + $\frac{5\pi}{4}$)

= cis $\frac{8\pi}{4}$ = cis 2π = cops 2π = cos 2π + isin2π

= 1 + io

Product = 1

7. Find all the values of $(-\sqrt{3}\ -\ \text{i})^{\frac{2}{3}}$

Answer

Let $-\sqrt{3}$ – i = r(cos$\theta + i \sin\theta$)

a = -$\sqrt{3}$	b = -1
a = -ve	b= -ve

r= $\sqrt{a^2 + b^2}$ = $\sqrt{3 + 1}$ =2

$\theta = tan^{-1} \left|\frac{b}{a}\right| - \pi = tan^{-1} \left(\frac{1}{\sqrt{3}}\right) - \pi = \frac{\pi}{6} - \pi = -\frac{5}{6}$

=-$\sqrt{3}$-I = 2(cos($\frac{-5\pi}{6}$) + i sin ($\frac{-5\pi}{6}$))

$= (-\sqrt{3} - i)^{\frac{2}{3}} = 2^{\frac{2}{3}}$ [cos($\frac{-5\pi}{6}$)+ isin($\frac{-5\pi}{6}$)]$^{2/3}$

=$2^{2/3}$[cos($\frac{-5\pi}{3}$)+isin($\frac{-5\pi}{3}$)]$^{1/3}$

=$2^{2/3}$[cos($2k\pi - \frac{-5\pi}{3}$)+isin($2k\pi - \frac{-5\pi}{3}$)]$^{1/3}$

= $2^{2/3}$ [cos(6k-5)$\frac{\pi}{9} + isin(6k-5)\frac{\pi}{9}$] k=0,1,2

Values are $2^{2/3}$cis($\frac{-5\pi}{9}$) ; $2^{2/3}$cis$\frac{\pi}{9}$; $2^{\frac{3}{3}} cis \frac{7\pi}{9}$

Exercise

Find all the values of ($\sqrt{3}$+i)$^{2/3}$ (June - 2007).

8. Solve: $x^4 - x^3 + x^2$-x +1 = 0 .

Solution: Given $x^4 - x^3 + x^2$-x +1 = 0 .

$\therefore \frac{x^5+1}{x+1} = 0$ { x ≠ -1.

x^5+ 1 = 0

x = (-1) $^{1/5}$

= (Cos π + i Sin π) $^{1/5}$

$= (\text{Cos} (2 k\pi + \pi) + i \text{ Sin } 2k\pi+\pi))^{1/5}$

$x = (\text{Cos} (2k+1)\frac{\pi}{5} + i \text{ Sin } (2k + 1) \frac{\pi}{5}$; $k = 0, 1,2,3,4.$

$\therefore Values\ are:$ $Cis\ \frac{\pi}{5}$; $Cis\ \frac{3\pi}{5}$; $Cis\ \frac{7\pi}{5}$; $Cis\ \frac{9\pi}{5}$ $\therefore$ {Except Cis π = -1.

9. Solve: $x^9 + x^5 - x^4 - 1 = 0$.

Solution: Given $x^9 + x^5 - x^4 - 1 = 0$.

$\therefore x^5 (x^4 + 1) - 1(x^4 + 1) = 0$.

$\therefore (x^5 - 1)(x^4 + 1) = 0$.

$(x^5 - 1) = 0$ (or) $(x^4 + 1) = 0$.

$x = 1^{1/5}$ (or) $x = -1^{1/4}$

<u>Values of x $= 1^{1/5}$.</u>

$x = (\text{Cos } 0 + i \text{ Sin } 0)^{1/5}$

$= (\text{Cos } 2k\pi + i \text{ Sin } 2k\pi)^{1/5}$

$= (\text{Cos} \frac{2k\pi}{5} + I \text{ Sin} \frac{2k\pi}{5})$; K = 0,1,2,3,4.

Values are ; $Cis\ 0$; $Cis\ \frac{2\pi}{5}$; $Cis\ \frac{4\pi}{5}$; $Cis\ \frac{6\pi}{5}$; $Cis\ \frac{8\pi}{5}$ ————> (1)

<u>Values of x $=(-1)^{1/4}$.</u>

$x = (\text{Cos } \pi + i \text{ Sin } \pi)^{1/4}$

$= (\text{Cos } 2k\pi + \pi)+(i \text{ Sin } 2k\pi+\pi)^{1/4}$

$x = (\text{Cos } (2k+1) \frac{\pi}{4} + \text{Sin } (2k+1) \frac{\pi}{4}$; K = 0,1,2,3,4.

Values are ; $Cis\ \frac{\pi}{4}$; $Cis\ \frac{3\pi}{4}$; $Cis\ \frac{5\pi}{4}$; $Cis\ \frac{7\pi}{4}$ ————> (2)

∴ Values in (1) and (2) are the roots of given equation.

Exercise: Solve: $x^7 + x^4 + x^3 + 1 = 0$.(June 2009.)